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# ON BIG BOSS GAMES*

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## 1. Introduction

In the literature, games with one big player generated from economic systems have been often studied in different contexts. These include in part: a market of an indivisible good with one seller and many buyers (von-Neumann and Morgenstern (1953), Kaneko (1976) and Tijs (1981)); a bankruptcy problem with one big claimant (O'Neill (1982), Aumann and Maschler (1985), and Curiel, Maschler and Tijs (1986)); a production economy with one landowner and many peasants (Shapley and Shubik (1967), Chetty, Dasgupta and Raghavan (1976), and Driessen and Tijs (1984)); an information good market with one possessor of information and many demanders (Muto, Potters and Tijs (1986), Muto (1986) and Nakayama (1986)). In most of these works, the cooperative game theoretic approach was taken, and solution concepts such as the core, the Shapley value, the nucleolus, and the $\tau$-value were studied to analyze the outcomes of these economic systems. Unfortunately, however, these solutions were studied separately in each of these works, and thus the relations of these solutions were rather ambiguous.

The aim of this paper is to consider a wide class of games containing these economic games, which we call big boss games, and to study comprehensively the solutions of these games. We clarify the relations of these solutions, and examine their economic implications. The remainder of the paper is organized as follows: In the next section, the definition of the big boss game as well as the definitions of solutions such as the core, stable sets, subsolutions, the Shapley value, the nucleolus and the $\tau$-value are given. In Sections 3 and 4, the properties of these solutions in big boss games are studied in detail. In Section 5, the economic systems mentioned above are reexamined from the standpoint of big boss games. The paper closes in Section 6 with a short summary of the results obtained in this paper. All proofs of propositions are given at the end of the paper as an appendix.

## 2. Big Boss Games

Let $(N, v)$ be a *cooperative game in characteristic function form, i.e.*, $N = \{1, 2, \ldots, n\}$ $(n \geqq 3)$ is a set of *players* and $v$ is a real valued function on $2^N$ (the set of all subsets of $N$) with $v(\phi) = 0$, called the *characteristic function*. For each subset of players (coalition) $S \subseteqq N$, $v(S)$ denotes the

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value or the worth of the coalition $S$. Hereafter the characteristic function $v$ itself is simply called a game. The set of all games (characteristic functions) $v$ with the set of players $N$ is denoted by $G^N$. In this paper, we will restrict our attention to a subset of $G^N$ consisting of all *monotonic* games. This set is denoted by $MG^N$. So, $v \in MG^N$ if and only if $v(S) \leqq v(T)$ whenever $S \subseteqq T$. Since $v(\phi) = 0$, this implies that $v(S) \geqq 0$ for all $S \subseteqq N$.

Let $v$ be a game in $MG^N$. $v$ is called a *big boss game* if there is one player, denoted by $i^*$, satisfying the following two conditions.

B1: $v(S) = 0 \quad \text{if } i^* \notin S$

and B2: $v(N) - v(S) \geqq \sum_{i \in N-S} (v(N) - v(N - \{i\})) \quad \text{if } i^* \in S.$

B1 implies that one player $i^*$ is very powerful, *i.e.*, coalitions not containing $i^*$ cannot get anything; and B2 implies that for every coalition not containing $i^*$, its contribution to the grand coalition is not less than the sum of the contributions of its players to the grand coalition. Hence, the weak players may increase their influences by forming coalitions. We here notice that a big boss game $v$ is *superadditive*, *i.e.*, $v(S) + v(T) \leqq v(S \cup T)$ whenever $S \cap T = \phi$, because of the monotonicity of $v$ and B1. We denote a set of all big boss games with the set of players $N$ and with the powerful player $i^*$ by $BBG^N_{i^*}$. We notice that $BBG^N_{i^*}$ forms a cone in $G^N$, *i.e.*, for all $v$, $w \in BBG^N_{i^*}$ and real numbers $a$, $b \geqq 0$, $av + bw$ is also in $BBG^N_{i^*}$. For simplicity, we take $i^* = 1$ and denote $BBG^N_1$ simply by $BBG^N$ throughout this paper.

In the following we study how well-known solution concepts behave in big boss games. Before starting the analysis, we briefly review their definitions.

Let $v$ be a game in $G^N$, and let $A(v)$ be the set of all *imputations* of the game $v$, *i.e.*, $A(v) = \{x \in R^n: \sum_{i \in N} x_i = v(N),\ x_i \geqq v(\{i\}) \text{ for all } i \in N\}$ where $R^n$ is the $n$-dimensional Euclidean space. Let $x, y \in A(v)$ and let $S \subseteqq N$ be a nonempty coalition. We say *x dominates y via S*, denoted by $x \text{ dom}_S\ y$, if (*i*) $x_i > y_i$ for all $i \in S$ and (ii) $\sum_{i \in S} x_i \leqq v(S)$. We simply say *x dominates y* (or *y is dominated by x*), denoted by $x \text{ dom } y$, if there is an $S \subseteqq N$ such that $x \text{ dom}_S\ y$. The negation of $x$ dom $y$ is denoted by $\sim x \text{ dom } y$. For an imputation $x$, let $\text{Dom}(x)$ be the set of imputations dominated by $x$, and for a set of imputations $B \subseteqq A$, let $\text{Dom}(B) = \bigcup_{x \in B} \text{Dom}(x)$. This means that $\text{Dom}(B)$ is a set of imputations which are dominated by at least one imputation in $B$.

**Core:** The *core* $C(v)$ of a game $v$ is the set of imputations which are not dominated by any other imputation, *i.e.*, $C(v) = \{x \in A(v): \sim y \text{ dom } x \text{ for all } y \in A(v)\}$. For each $x \in A(v)$ and $S \subseteqq N$, let $e(S, x) = v(S) - \sum_{i \in S} x_i$. $e(S, x)$ is called the *excess of S with respect to x*. It is known that the core $C(v)$ is given by $C(v) = \{x \in A(v): e(S, x) \leqq 0 \text{ for all } S \subseteqq N\}$ if the game $v$ is superadditive.

**Stable sets:** A *stable set* of a game $v$ is a nonempty set $K$ of imputations satisfying the properties (i) for all $x, y \in K$, $\sim x$ dom $y$ and $\sim y$ dom $x$ (internal stability), and (ii) for all $z \in A(v) - K$, there is an imputation $x \in K$ such that $x$ dom $z$ (external stability). The core always satisfies the internal stability, and thus if the core is nonempty and at least one stable set exists, then the core is included in all stable sets. When the core also satisfies the external stability and thus when it is a stable set, it is called the *stable core.* In this case, the stable core is the unique stable set. The stable core has a very strong stability since it is not dominated by any imputation, and furthermore every imputation outside it, is dominated by some imputation belonging to it. It is known that if a game $v$ is *convex, i.e.*, for all $i \in N$, $v(S) - v(S - \{i\}) \leqq v(T) - v(T - \{i\})$ for all $S$, $T \subseteqq N$ with $i \in S \subseteqq T$, then the core is the stable core.

**Subsolution:** A *subsolution* of a game $v$ is a nonempty set $L$ of imputations satisfying the following properties: (i) for all $x, y \in L$, $\sim x$ dom $y$ and $\sim y$ dom $x$ (internal stability); (ii) if $x \in L$ and $y$ dom $x$, then $y \in \mathrm{Dom}(L)$; and (iii) if $x \notin L \cup \mathrm{Dom}(L)$, then there is an imputation $y \notin L \cup \mathrm{Dom}(L)$ such that $y$ dom $x$. It is known that if the core is nonempty, then a subsolution exists, and further that the intersection of all subsolutions is also a subsolution; the intersection is called the *super core.* The core always satisfies (i) and (ii). When the core satisfies also (iii), it is the super core. If the core is the super core, it has a stronger stability than the usual core in the sense that all imputations outside the core and further not dominated by the core must be dominated by some imputation having the same property. Hence, in the region $A(v) - (C(v) \cup \mathrm{Dom}(C(v)))$, there is no stable subset of imputations.

**Shapley value:** Let $\sigma$ be the set of all orders of players $1, 2, \ldots, n$. Take a player $i \in N$ and an order $\sigma \in \sum$, and suppose $i = \sigma(k)$. Let us denote by $m(\sigma, i)$ the player $i$'s contribution in $\sigma$, *i.e.*, $m(\sigma, i) = v(\{\sigma(1), \ldots, \sigma(k-1), \sigma(k)\}) - v(\{\sigma(1), \ldots, \sigma(k-1)\})$. Then the *Shapley value* $\phi(v)$ of $v$ is defined by $\phi(v) = (\phi_i(v))_{i \in N}$ where $\phi_i(v) = (\sum_{\sigma \in \sum} m(\sigma, i))/n!$ for all $i \in N$. It is known that for each $i \in N$, $\phi_i(v)$ is given by $\phi_i(v) = (1/n!) \sum_{S \subseteqq N - \{i\}} s!(n-s-1)!(v(S \cup \{i\}) - v(S))$ where $s = |S|$ denotes the number of players in $S$.

**Nucleolus:** For each $x \in A(v)$, let $\theta(x)$ be the $2^n$-tuple whose components are the excesses $e(S, x)$, $S \subseteqq N$, arranged in nonincreasing order, *i.e.*, $\theta_i(x) \geqq \theta_j(x)$ whenever $1 \leqq i < j \leqq 2^n$. For $x, y \in A(v)$, we say that $x$ *is more acceptable than* $y$ if there exists an integer $1 \leqq k \leqq 2^n$ such that $\theta_i(x) = \theta_i(y)$ for all $i$ with $1 \leqq i < k$ and $\theta_k(x) < \theta_k(y)$. The *nucleolus* of $v$ is the set of imputations $x$ such that there is no imputation being more acceptable than $x$. It is known that the nucleolus is not empty and consists of a single imputation. In the following the unique nucleolus element of $v$ is denoted by $\nu(v)$.

**$\tau$-Value:** For each $i \in N$, let $M_i(v) = v(N) - v(N - \{i\})$, *i.e.*, $M_i(v)$ is the player $i$'s marginal contribution to the grand coalition, and let $M(v) = (M_i(v))_{i \in N}$. For each $i \in N$ and each coalition $S \subseteqq N$ with $S \ni i$, let $R(S, i) = v(S) - \sum_{j \in S - \{i\}} M_j(v)$, $m_i(v) = \max\{R(S, i) : S \subseteqq N, S \ni i\}$, and $m(v) = (m_i(v))_{i \in N}$. In case a game $v$ has a nonempty core, the *$\tau$-value* $\tau(v) = (\tau_i(v))$ is given by the convex combination of $M(v)$ and $m(v)$ satisfying $\sum_{i \in N} \tau_i(v) = v(N)$.

The Shapley value, the nucleolus, and the $\tau$-value possess a symmetry property in the sense that if players $i, j$ are symmetric in a game $v$, then in each of these three solutions, they get equal payoffs. Here two players $i, j \in N$ are said to be *symmetric* in the game $v$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteqq N - \{i, j\}$.

For details of these definitions, the readers may refer to Gillies (1953) (the core), von Neumann and Morgenstern (1953) (stable sets), Roth (1976) (subsolutions), Shapley (1953) (the Shapley value), Schmeidler (1969) (the nucleolus), and Tijs (1981) (the $\tau$-value).

## 3. The core, Stable Sets and Subsolutions

Before proceeding with the analysis, we calculate $M(v)$ and $m(v)$ defined in the previous section.

**Proposition 3.1:** Suppose $v \in BBG^N$. Then the following statements hold:

(1) $M_i(v) \geqq 0$ for all $i \in N - \{1\}$,

(2) $M_1(v) = v(N)$,

(3) $m(v) = (v(N) - \sum_{i \in N-\{1\}} M_i(v), 0, \ldots, 0)$,

and (4) $M(v) \geqq m(v) \geqq 0$.

As shown below, the core of a big boss game is always nonempty, and it takes a simple parallelotope structure. We first define a set $H(v)$ by

$$H(v) = \{x \in R^n : \sum_{i \in N} x_i = v(N),\ 0 \leqq x_i \leqq M_i(v) \text{ for all } i \in N - \{1\}\}$$

It easily follows that $H(v)$ is nonempty whenever $v$ is in $MG^N$, since from Proposition 3.1(1), $(v(N), 0, \ldots, 0)$ is always contained in $H(v)$. The following theorem shows that in a big boss game $v$ the core $C(v)$ coincides with $H(v)$, and that further in the space of $MG^N$ the converse also holds.

**Theorem 3.2:** Let $v \in MG^N$. Then $v \in BBG^N$ if and only if $C(v) = H(v)$.

**Proof:** Necessity: Take a big boss game $v$. To show $C(v) \subseteqq H(v)$, take $x \in C(v)$. $\sum_{i \in N} x_i = v(N)$ is clearly satisfied. Take $i \in N - \{1\}$. Then since $v(\{i\}) = 0$ (from B1) and $x_i \geqq v(\{i\})$, $x_i \geqq 0$ holds. Further since $\sum_{j \in N-\{i\}} x_j \geqq v(N - \{i\})$, we have $x_i = v(N) - \sum_{j \in N-\{i\}} x_j \leqq v(N) - v(N - \{i\}) = M_i(v)$. Therefore $C(v) \subseteqq H(v)$ follows. To show the reverse inclusion, take $x \in H(v)$. It suffices to show that $\sum_{i \in S} x_i \geqq v(S)$ for all $S \subseteqq N$. Take $S \subseteqq N$. If $1 \notin S$, then $\sum_{i \in S} x_i \geqq v(S)$ is trivial from B1. Suppose $1 \in S$. Then we obtain $\sum_{i \in S} x_i = v(N) - \sum_{i \in N-S} x_i \geqq v(N) - \sum_{i \in N-S} M_i(v) \geqq v(S)$ where inequalities follow from the fact $x_i \leqq M_i(v)$ for all $i \in N - \{1\}$ and B2. Therefore $C(v) \supseteqq H(v)$ holds.

Sufficiency: Take $x = (v(N), 0, \ldots, 0) \in H(v)$. Then $x$ must be in the core since $C(v) = H(v)$. Take $S \subseteqq N - \{1\}$. Then $\sum_{i \in S} x_i = 0 \geqq v(S)$, and thus we obtain $v(S) = 0$ from the monotonicity of $v$. Hence B1 holds. Now take $x = (v(N) - \sum_{i \in N-\{1\}} M_i(v), M_2(v), \ldots, M_n(v)) \in H(v)$. Take $S \subseteqq N$ with $1 \in S$. Then $\sum_{i \in S} x_i = v(N) - \sum_{i \in N-\{1\}} M_i(v) + \sum_{i \in S-\{1\}} M_i(v) = v(N) - \sum_{i \in N-S} M_i(v)$. Since $C(v) = H(v)$, we must have $\sum_{i \in S} x_i \geqq v(S)$. Therefore B2 holds. (Q.E.D.)

Therefore the core of an $n$-person big boss game is in general an $(n-1)$-dimensional parallelotope with $2^{n-1}$ vertices of the form $(u_1, u_2, \ldots, u_n)$ where $u_i = 0$ or $M_i(v)$ for $i = 2, \ldots, n$ and $u_1 = v(N) - \sum_{i \in N-\{1\}} u_i$. Hence, in the core, each of the weak players $2, \ldots, n$ gains at most his marginal contribution to the grand coalition; but it might also occur that they are completely exploited by the big boss. Figure 1 illustrates the cores of 3-person and 4-person big boss games.

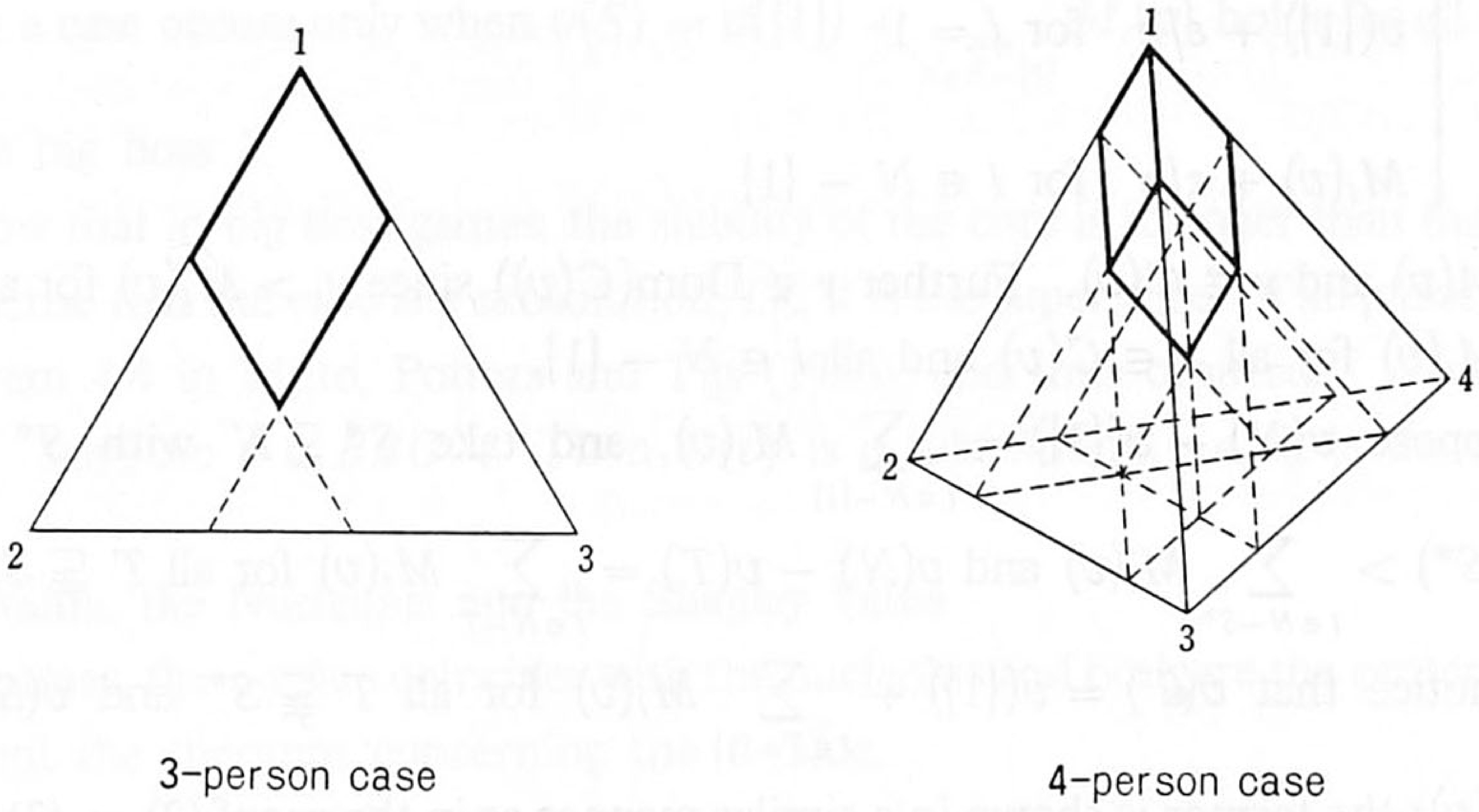


**Figure 1** The cores of 3-person and 4-person big boss games

We hereupon make a remark on the core cover defined by Tijs (1981) and its relation to $H(v)$. The *core cover* $CC(v)$ is given by

$$CC(v) = \{x \in R^n\colon \sum_{i \in N} x_i = v(N),\ m_i(v) \leqq x_i \leqq M_i(v) \text{ for all } i \in N\}.$$

The following proposition shows that in big boss games $CC(v)$ coincides with $H(v)$.

**Proposition 3.3:** Suppose $v \in BBG^N$. Then $H(v) = CC(v)$.

Hence together with Theorem 3.2 we obtain that the core and the core cover both coincide with $H(v)$.

We now examine the stability of the core, by analyzing its relations to stable sets and subsolutions. As shown below, in big boss games, the core itself is a stable set (*i.e.*, it is the stable core), if and only if the game is convex. We first characterize the convexity in big boss games.

**Proposition 3.4:** Suppose $v \in BBG^N$. Then the following assertions are equivalent:

(1) $v$ is convex;

(2) equality holds in every inequality in B2, *i.e.*,

$$v(N) - v(S) = \sum_{i \in N-S} M_i(v) \text{ for all } S \subseteqq N \text{ with } S \ni 1;$$

(3) $v(S) = v(\{1\}) + \sum_{i \in S-\{1\}} M_i(v)$ for all $S \subseteqq N$ with $S \ni 1$;

and (4) $v(S) - v(S - \{i\}) = M_i(v)$ for all $S \subseteqq N$ with $S \supsetneqq \{1\}$ and for all $i \in S - \{1\}$.

Now we prove the theorem.

**Theorem 3.5:** Suppose $v \in BBG^N$. Then the core $C(v)$ is a stable set, *i.e.*, it is the stable core, if and only if $v$ is convex.

**Proof:** Since it is well-known that the sufficiency holds for all games in $G^N$ (Shapley (1971)), we will show the necessity. Suppose $v \in BBG^N$ and $v$ is not convex. Then from Proposition 3.4, there must exist at least one $S \subseteqq N$ with $S \ni 1$ satisfying $v(N) - v(S) > \sum_{i \in N-S} M_i(v)$. First consider the case where $v(N) - v(\{1\}) > \sum_{i \in N-\{1\}} M_i(v)$. Let $\varepsilon = v(N) - (v(\{1\}) + \sum_{i \in N-\{1\}} M_i(v)) > 0$, and define an $n$-dimensional vector $y$ by

$$y_i = \begin{cases} v(\{1\}) + \varepsilon/n & \text{for } i = 1 \\ M_i(v) + \varepsilon/n & \text{for } i \in N - \{1\} \end{cases}$$

Then $y \in A(v)$ and $y \notin C(v)$. Further $y \notin \mathrm{Dom}(C(v))$ since $y_i > M_i(v)$ for all $i \in N - \{1\}$ and $x_i \leqq M_i(v)$ for all $x \in C(v)$ and all $i \in N - \{1\}$.

Now suppose $v(N) - v(\{1\}) = \sum_{i \in N-\{1\}} M_i(v)$, and take $S^* \subseteqq N$ with $S^* \ni 1$ satisfying $v(N) - v(S^*) > \sum_{i \in N-S^*} M_i(v)$ and $v(N) - v(T) = \sum_{i \in N-T} M_i(v)$ for all $T \subsetneqq S^*$ with $T \ni 1$. We here notice that $v(T) = v(\{1\}) + \sum_{i \in T-\{1\}} M_i(v)$ for all $T \subsetneqq S^*$ and $v(S^*) < v(\{1\}) + \sum_{i \in S^*-\{1\}} M_i(v)$; the former is shown in a similar manner as in the proof (2) $\Rightarrow$ (3) of Proposition 3.4, and the latter follows from $v(N) - v(\{1\}) = \sum_{i \in N-\{1\}} M_i(v)$ and $v(N) - v(S^*) > \sum_{i \in N-S^*} M_i(v)$. We further note that $2 \leqq s^* = |S^*| \leqq n - 2$. We next show that $M_i(v) > 0$ for all $i \in S^* - \{1\}$. Suppose not, and take $i \in S^* - \{1\}$ with $M_i(v) = 0$. Then from the definition of $S^*$, we have $v(N) - v(S^* - \{i\}) = \sum_{j \in N-S^*} M_j(v) + M_i(v) < v(N) - v(S^*) + M_i(v)$. Hence $v(S^*) < v(S^* - \{i\}) + M_i(v)$. If $M_i(v) = 0$, then we obtain $v(S^*) < v(S^* - \{i\})$ which contradicts the monotonicity of $v$. Noting this positivity of $M_i(v)$ and the fact $2 \leqq s^* \leqq n - 2$, define an $n$-dimensional vector $y$ by

$$y_i = \begin{cases} v(\{1\}) + (s^* - 2)\varepsilon & \text{for } i = 1 \\ M_i(v) - \varepsilon & \text{for all } i \in S^* - \{1\} \\ M_i(v) + \varepsilon & \text{for one } i^* \in N - S^* \\ M_i(v) & \text{for all } i \in (N - S^*) - \{i^*\} \end{cases}$$

where $\varepsilon$ is a sufficiently small positive number such that $\varepsilon < \min\,(v(\{1\}) + \sum_{i \in S^*-\{1\}} M_i(v) - v(S^*), (\{M_i(v)\}_{i \in S^*-\{1\}})$. Since $v(\{1\}) + \sum_{i \in N-\{1\}} M_i(v) = v(N)$, we have $y \in A(v)$. Further $y \notin C(v)$ since $y_{i^*} > M_{i^*}(v)$. Now suppose that there is $x \in C(v)$ such that $x$ doms y. From B1, $S$ must contain player 1. Since $y_i \geqq M_i(v)$ for all $i \in N - S^*$ and $\sum_{i \in S^*} y_i > v(S^*)$ from the definition of $\varepsilon$ and $y$, we must have $S \subsetneqq S^*$. It holds, however, that $\sum_{i \in T} y_i \geqq v(T)$ for all $T \subsetneqq S^*$ with $T \ni 1$. In fact, for such $T$, $\sum_{i \in T} y_i = v(\{1\}) + (s^* - 2)\varepsilon + \sum_{i \in T-\{1\}} M_i(v) -$

$(t-1)\varepsilon = v(\{1\}) + \sum_{i \in T-\{1\}} M_i(v) + (s^* - t - 1)\varepsilon$ where $t = |T|$. Since $v(\{1\}) + \sum_{i \in T-\{1\}} M_i(v) = v(T)$ and $t \leqq s^* - 1$, we obtain $\sum_{i \in T} y_i \geqq v(T)$. Therefore we obtain that $x$ doms $y$ never occurs in case $S \subsetneqq S^*$, and thus $x \notin \mathrm{Dom}(C(v))$ follows. (Q.E.D.)

Therefore, together with Proposition 3.4, we may claim that in big boss games the cases in which the stable core exists, *i.e.*, the core has a strong stability, are restricted. More precisely speaking, such a case occurs only when $v(S) = v(\{1\}) + \sum_{i \in S-\{1\}} M_i(v)$ holds for all coalitions $S$ containing the big boss 1.

We now show that in big boss games, the stability of the core is stronger than that in general games in the sense that the core is a subsolution, *i.e.*, it is the super core. The proof is similar to that of Theorem 4.4 in Muto, Potters and Tijs (1986), and thus omitted.

**Theorem 3.6:** Suppose $v \in BBG^N$. Then $C(v)$ is a subsolution, *i.e.*, it is the super core.

## 4. The $\tau$-Value, the Nucleolus and the Shapley Value

In big boss games, the $\tau$-value coincides with the nucleolus and both are the center of the core. We first present the theorem concerning the $\tau$-value.

**Theorem 4.1:** Let $v \in BBG^N$. Then the $\tau$-value $\tau(v)$ of $v$ is given by $\tau(v) = (v(N) - (\sum_{i \in N-\{1\}} M_i(v))/2, M_2(v)/2, \ldots, M_n(v)/2)$, *i.e.*, $\tau(v)$ is the center of the core $C(v)$.

**Proof:** Since $\tau(v)$ is the convex combination of $M(v)$ and $m(v)$ satisfying $\sum_{i \in N} \tau_i(v) = v(N)$, the theorem easily follows from Proposition 3.1(2), (3). (Q.E.D.)

Further, the following theorem shows that the $\tau$-value is also the nucleolus.

**Theorem 4.2:** Let $v \in BBG^N$. Then the nucleolus $\nu(v)$ of $v$ is given by $\nu(v) = \tau(v) = (v(N) - (\sum_{i \in N-\{1\}} M_i(v))/2, M_2(v)/2, \ldots, M_n(v)/2)$.

**Proof:** Though this theorem may be proved by using the Kohlberg's condition (Kohlberg (1971)) in a similar manner as in Muto, Potters and Tijs (1986, the proof of Theorem 5.3), we present a direct proof in the following. Throughout the proof, we assume without loss of generality $M_2(v) \leqq \ldots \leqq M_n(v)$. Recall that $M_2(v) \geqq 0$ (Proposition 3.1(1)). We first prove two claims.

**Claim 1:** For each $i \in N - \{1\}$, $e(\{i\}, \tau(v)) = e(N - \{i\}, \tau(v)) = -M_i(v)/2$.

**Proof of Claim 1:** Take $i \in N - \{1\}$. Then from B1 and Theorem 4.1, $e(\{i\}, \tau(v)) = v(\{i\}) - M_i(v)/2 = -M_i(v)/2$. Further from the definition of $M_i(v)$ and Theorem 4.1, $e(N - \{i\}, \tau(v)) = v(N - \{i\}) - (v(N) - (\sum_{j \in N-\{1\}} M_j(v))/2) - (\sum_{j \in N-\{1,i\}} M_j(v))/2 = -M_i(v) + (\sum_{j \in N-\{1\}} M_j(v))/2 - (\sum_{j \in N-\{1,i\}} M_j(v))/2 = -M_i(v)/2$. (Q.E.D.)

Therefore from the assumption $M_2(v) \leqq \ldots \leqq M_n(v)$, we have $e(N,\tau(v)) = e(\phi, \tau(v)) = 0 \geqq e(\{2\}, \tau(v)) = e(N - \{2\}, \tau(v)) \geqq \ldots \geqq e(\{2\}, \tau(v)) = e(\{n\}, \tau(v)) = e(N - \{n\}, \tau(v))$.

**Claim 2:** For each $i \in N - \{1\}$, if $e(S, \tau(v)) > e(\{i\}, \tau(v)) = e(N - \{i\}, \tau(v)) = -M_i(v)/2$, then $S \supseteqq \{1, i, i+1, \ldots, n\}$ or $S \subseteqq \{2, \ldots, i-1\}$.

**Proof of Claim 2:** Take $i' \in N - \{1\}$ with $e(S, \tau(v)) > e(\{i'\}, \tau(v))$. Suppose $1 \in S$. Then from B2, we obtain $e(S, \tau(v)) = v(S) - (v(N) - (\sum_{i \in N-\{1\}} M_i(v))/2) - (\sum_{i \in S-\{1\}} M_i(v))/2 = v(S) - v(N) + (\sum_{i \in N-S} M_i(v))/2 \leqq -(\sum_{i \in N-S} M_i(v))/2$. If $S \not\supseteqq \{1, i'i' + 1, \ldots, n\}$, then $N - S$ contains at least one $i \geqq i'$. Therefore we have a contradiction $e(S, \tau(v)) \leqq -(\sum_{i \in N-S} M_i(v))/2 \leqq -M_{i'}(v)/2 = e(\{i'\}, \tau(v))$. Next suppose $1 \notin S$. Then from B1, $e(S, \tau(v)) = -(\sum_{i \in S} M_i(v))/2$. If $S \not\subseteqq \{2, \ldots, i' - 1\}$, then $S$ contains at least one $i \geqq i'$. We thus have a contradiction $e(S, \tau(v)) = -(\sum_{i \in S} M_i(v))/2 \leqq -M_{i'}(v)/2 = e(\{i'\}, \tau(v))$.

(Q.E.D.)

Now take any $x \in A(v) - \{\tau(v)\}$, and let $i^* = \min\{i \in N - \{1\} : x_i \neq \tau_i(v)\}$. Since $x_i = \tau_i(v) = -M_i(v)/2$ for all $i = 2, \ldots, i^* - 1$, we have $\sum_{i \in S} x_i = \sum_{i \in S} \tau_i(v)$ for all $S \subseteqq N$ with $S \supseteqq \{1, i^*, i^* + 1, \ldots, n\}$ or $S \subseteqq \{2, \ldots, i^* - 1\}$. Therefore $e(S, x) = e(S, \tau(v))$ for all $S \supseteqq N$ such that $S \supseteqq \{1, i^*, i^* + 1, \ldots, n\}$ or $S \subseteqq \{2, \ldots, i^* - 1\}$. Consider a coalition $S$ with $e(S, \tau(v)) > -M_{i^*}(v)/2$. Then from Claim 2, we have $S \supseteqq \{1, i^*, i^* + 1, \ldots, n\}$ or $S \subseteqq \{2, \ldots, i^* - 1\}$. For these coalitions, we have $e(S, x) = e(S, \tau(v))$ as shown above. Now one of the excesses $e(\{i^*\}, x)$ or $e(N - \{i^*\}, x)$ is greater than $e(\{i^*\}, \tau(v)) = e(N - \{i^*\}, \tau(v)) = -M_{i^*}(v)/2$. In fact, if $x_{i^*} < M_{i^*}(v)/2$, then $e(\{i^*\}, x) = -x_{i^*} > -M_{i^*}(v)/2$, and if $x_{i^*} > M_{i^*}(v)/2$, then $e(N - \{i^*\}, x) = v(N - \{i^*\}) - (v(N) - x_{i^*}) = -M_{i^*}(v) + x_{i^*} > -M_{i^*}(v)/2$. Therefore we obtain that $\tau(v)$ is more acceptable than $x$, which implies that $\tau(v)$ is the nucleolus. (Q.E.D.)

From Theorems 4.1 and 4.2, we notice that the $\tau$-value and the nucleolus possess the additivity property in big boss games. Namely, for all $v$, $w \in BBG^N$, $\tau(v + w) = \tau(v) + \tau(w)$ and $\nu(v + w) = \nu(v) + \nu(w)$. ($v + w$ is in $BBG^N$ since $BBG^N$ forms a cone in $G^N$ as mentioned in Section 2.)

Now let us examine the Shapley value. While in the $\tau$-value and the nucleolus, each of the weak players $2, \ldots, n$ gets exactly half of his marginal contribution to the grand coalition and the rest goes to the big boss 1, in the Shapley value, the big boss generally gets less.

**Theorem 4.3:** Let $v \in BBG^N$, and $\phi(v)$ be the Shapley value of $v$. Then $\phi_1(v) \leqq v(N) - (\sum_{i \in N-\{1\}} M_i(v))/2$.

**Proof:** Let $\sum$ be a set of all orders of players $1, 2, \ldots, n$. Recalling $m(\sigma, i)$ denotes player $i$'s contribution in an order $\sigma$, the Shapley value of player $i$ is given by $(\sum_{\sigma \in \sum} m(\sigma, i))/n!$. Now take an order $\sigma$, and let $1 = \sigma(k)$, $S = \{\sigma(1), \ldots, \sigma(k - 1)\}$, and $T = \{\sigma(k + 1), \ldots, \sigma(n)\}$. Take another order $\sigma'$ in which the $j$th player in $\sigma$ is in the $(n + 1 - j)$th position, i.e., $\sigma(j) = \sigma'(n + 1 - j)$. Hence in $\sigma'$, the player 1 is in the $(n + 1 - k)$th position, $T$ consists of the first $(n - k)$ players, and $S$ consists of the last $(k - 1)$ players. From B1, we obtain $m(\sigma, i) = 0$ for all $i \in S$. Further from B2, $\sum_{i \in T} m(\sigma, i) = v(N) - v(S \cup \{1\}) \geqq$

$\sum_{i \in N-(S \cup \{1\})} M_i(v) = \sum_{i \in T} M_i(v)$. Since $\sum_{i \in N} m(\sigma, i) = v(N)$, we obtain $m(\sigma, 1) \leqq v(N) - \sum_{i \in T} M_i(v)$. Similarly, in $\sigma'$, we get $m(\sigma', 1) \leqq v(N) - \sum_{i \in S} M_i(v)$. Therefore $m(\sigma, 1) + m(\sigma', 1) \leqq 2v(N) - \sum_{i \in N-\{1\}} M_i(v)$. Since there are $n!/2$ such pairs $\{\sigma, \sigma'\}$, the desired inequality $\phi_1(v) \leqq v(N) - (\sum_{i \in N-\{1\}} M_i(v))/2$ follows. (Q.E.D.)

If the game is symmetric with respect to the weak players $2, \ldots, n$, *i.e.*, every two of these players are symmetric, then they get equal payoffs in each of the $\tau$-value, the nucleolus, and the Shapley value as mentioned in Section 2. Thus each of the weak players generally gets more in the Shapley value. Here we note that the game $v \in BBG^N$ is symmetric with respect to the players $2, \ldots, n$, if and only if for every $S \subseteqq N$ with $S \ni 1$, $v(S)$ depends only on the number of players in $S$, and thus all $M_i(v)$, $i \in N - \{1\}$, have the same value.

**Corollary 4.4:** Let $v \in BBG^N$. If $v$ is symmetric with respect to the players $2, \ldots, n$, then $\phi_i(v) \geqq M_i(v)/2$ for all $i \in N - \{1\}$.

**Proof:** This follows from the symmetry of $v$ and Theorem 4.3. (Q.E.D.)

Our next interest is when the big boss 1 gets the same payoff in the Shapley value and in the $\tau$-value (and thus in the nucleolus). The following theorem answers the question.

**Theorem 4.5:** Let $v \in BBG^N$. Then the following assertions are equivalent:

(1) $\phi_1(v) = \tau_1(v) = \nu_1(v)$;

(2) $\phi(v) = \tau(v) = \nu(v)$;

and (3) $v$ is convex.

**Proof:** (1)$\Rightarrow$(3): Reviewing the proof of Theorem 4.3, we must have $v(N) - v(S) = \sum_{i \in N-S} M_i(v)$ for all $S \subseteqq N$ with $S \ni 1$ whenever (1) holds. Therefore from Proposition 3.4 the convexity of $v$ follows.

(3)$\Rightarrow$(2): We first note that $v(S) - v(S - \{i\}) = M_i(v)$ for all $S \subseteqq N$ with $S \supsetneqq \{1\}$ and for all $i \in S - \{1\}$. (Recall Proposition 3.4.) Take player $i \in N - \{1\}$. Let $\sigma$ be an order of players $1, \ldots, n$, and suppose $i = \sigma(k)$. Let $S = \{\sigma(1), \ldots, \sigma(k-1)\}$ and $T = \{\sigma(k+1), \ldots, \sigma(n)\}$. Similarly as in the proof of Theorem 4.3, let $\sigma'$ be the order generated from $\sigma$ in a manner so that the order of the players is completely reversed. Then in $\sigma$, $i$'s contribution $m(\sigma, i)$ is given by

$$m(\sigma, i) = \begin{cases} v(S \cup \{i\}) - v(S) & \text{if } 1 \in S \\ 0 & \text{if } 1 \notin S. \end{cases}$$

In $\sigma'$, $m(\sigma', i)$ is given by

$$m(\sigma', i) = \begin{cases} 0 & \text{if } 1 \in S \text{ and thus } 1 \notin T \\ v(T \cup \{i\}) - v(T) & \text{if } 1 \notin S \text{ and thus } 1 \in T. \end{cases}$$

Therefore noticing the remark mentioned at the beginning of this proof, we obtain $m(\sigma, i) + m(\sigma', i) = M_i(v)$. Since there are $n!/2$ such pairs $\{\sigma, \sigma'\}$, we obtain

$\phi_i(v) = M_i(v)/2 = \nu_i(v) = \tau_i(v)$. Thus $\phi(v) = \tau(v) = \nu(v)$ follows.

(2)$\Rightarrow$(1): This clearly holds. (Q.E.D.)

From this theorem, we see that if the big boss 1 gets the same payoff in the Shapley value and in the $\tau$-value (and in the nucleolus), then all other weak players also get the same payoffs in these solutions. We further notice from Proposition 3.4 that in such games, $v$ is given by $v(S) = v(\{1\}) + \sum_{i \in S-\{1\}} M_i(v)$ for all $S \subseteqq N$ with $S \ni 1$. Recall in such games the core $C(v)$ is the stable core. Except in these games, the big boss is strictly worse off in the Shapley value than in the $\tau$-value or the nucleolus. Therefore, in total, the Shapley value is strictly more favorable to weak players. It is not necessarily true, however, that each of the weak players is better off in the Shapley value as the following example reveals.

**Example 4.1:** $N = \{1, 2, 3, 4\}, v(\{1, 2, 3, 4\}) = 8, v(\{1, 2, 3\}) = 6, v(\{1, 2, 4\}) = 4, v(\{1, 3, 4\}) = 6$, $v(\{1, 3\}) = 3$, $v(\{1, 4\}) = 2$, and $v(S) = 0$ for all other $S \subseteqq N$.

Since $M_1(v) = 8$, $M_2(v) = 2$, $M_3(v) = 4$, and $M_4(v) = 2$, it is easily seen that $v$ is a big boss game. Therefore from Theorems 4.1 and 4.2, the $\tau$-value and the nucleolus are given by $\tau(v) = \nu(v) = (4, 1, 2, 1)$. We get, through straightforward calculation, $\phi(v) = (45/12, 11/12, 25/12, 15/12)$. We see that $\phi_3(v) > \tau_3(v)$, $\phi_4(v) > \tau_4(v)$, but $\phi_2(v) < \tau_2(v)$.

One of the sufficient conditions which ensure that $\phi_i(v) \geqq \tau_i(v)$ for all $i \in N - \{1\}$ is the following.

B2*: For all $S$, $T \subseteqq N$ such that $1 \in S \subseteqq T$, $v(T) - v(S) \geqq \sum_{i \in T-S} M_i(v)$.

This condition implies that for every coalition not containing the big boss 1, its contribution to its proper super set containing player 1 is not less than the sum of the marginal contribution of its players to the grand coalition. We easily notice that B2* implies B2. In fact, if we put $T = N$ in B2*, then we get B2. We further notice that Example 4.1 does not satisfy B2* since $v(\{1, 3\}) - v(\{1\}) = 3 < 4 = M_3(v)$. As shown below, B2* is equivalent to the following condition B2**.

B2**: For all $S \subseteqq N$ with $S \supsetneqq \{1\}$, $v(S) - v(S - \{i\}) \geqq M_i(v)$ for all $i \in S - \{1\}$.

B2** says that for each of the weak players, his contribution to each of the coalitions containing player 1 is not less than his marginal contribution to the grand coalition.

**Proposition 4.6:** B2* and B2** are equivalent for all games in $G^N$.

We call a game $v \in MG^N$ satisfying B1 and B2* (or B2**) a *strong big boss game*, and denote a set of all strong big boss games by $SBBG^N$. Now we have the following theorem.

**Theorem 4.7:** Let $v \in SBBG^N$, and $\phi(v)$ be the Shapley value of $v$. Then $\phi_i(v) \geqq M_i(v)/2$ for all $i \in N - \{1\}$.

**Proof:** Using the fact that $v$ satisfies B2**, this theorem is proved in a similar manner as in the proof (3)$\Rightarrow$(2) of Theorem 4.5. (Q.E.D.)

## 5. Economic Applications of Big Boss Games

In this section, we analyze the following economic systems, using the results obtained in previous sections: (1) an indivisible good market with one seller and many buyers; (2) a bankruptcy problem with one big claimant; (3) a production economy with one landowner and many peasants; (4) a market of an information good with symmetric externalities; and (5) a market of an information good with respect to a new product.

### 5.1 An Indivisible Good Market with One Seller and Many Buyers

We consider a market consisting of $n$ traders and two kinds of goods; an indivisible good and money. Let $N = \{1, 2, \ldots, n\}$ be a set of traders where 1 is a seller and $2, \ldots, n$ are buyers. The seller 1 initially holds $w$ units of the indivisible good where $w$ is a positive integer. Each of the buyers does not hold any of the good, *i.e.*, the buyer $i$'s initial holding of the good is 0. Each trader $i$ has a utility function $u_i$ measured in terms of money. $u_i(k)$ denotes $i$'s utility for $k$ units of the good. As for the seller 1, we assume $u_1(0) = 0$. We further assume $u_1(k) \geqq u_1(k-1)$ for all $k \geqq 1$, and $u_1(k) - u_1(k-1) \geqq u_1(k+1) - u_1(k)$ for all $k \geqq 1$; the former implies the monotone nondecreasingness of $u_1$, and the latter implies the concavity of $u_1$. Letting $a_k = u_1(w-k+1) - u_1(w-k)$ for $k = 1, \ldots, w$, we have $0 \leqq a_1 \leqq a_2 \leqq \ldots \leqq a_w$. For each buyer $i = 2, \ldots, n$, we assume $u_i(0) = 0$ and $u_i(k) = h_i$ for all $k \geqq 1$, where $h_i$ is a positive number. This expresses that each buyer demands essentially only one unit of the good. We assume for simplicity that $h_2 \geqq h_3 \geqq \ldots \geqq h_n$.

The characteristic function $v(S)$ is given by

(1) if $S \not\ni 1$, then $v(S) = 0$,

(2) if $S = \{1\}$, then $v(S) = \sum_{k=1}^{w} a_k$,

and (3) if $S \supsetneqq \{1\}$, then letting $S = \{1\} \cup \{i(1), \ldots, i(s-1)\}$ ($s$ is the number of players in $S$ and $i(1) < \ldots < i(s-1)$), and $m(S) = \max\{t(\text{integer}) : 1 \leqq t \leqq \min(s-1, w),\ h_{i(t)} > a_t\}$, $v(S) = h_{i(1)} + \ldots + h_{i(m(S))} + a_{m(S)+1} + \ldots + a_w$. For convenience, we let $m(S) = 0$ in case $h_{i(1)} \leqq a_1$. Note that $m(S)$ is the units of the goods traded in the coalition $S$.

In the simplest case with $n = 3$ and $w = 1$, von Neumann and Morgenstern (1953) studied the core and the stable set, and Tijs (1981) described the $\tau$-value. Kaneko (1976) obtained the core of the game with general $n$ and $w$ as a corollary of his study of more general cases including a market with more than one seller.

Now we will show that this game is a strong big boss game. The monotonicity of $v$ and B1 are clearly satisfied. Further, we may show in the following manner that $v$ also satisfies B2**. We first notice that for all coalitions $S \supsetneqq \{1\}$, and for all $i \in S - \{1\}$,

$$v(S) - v(S - \{i\}) = \begin{cases} 0 & \text{if } i > i(m(S)) \\ h_i - \max(h_{i(m(S))+1}, a_{m(S)}) & \text{if } i \leqq i(m(S)) \end{cases}$$

$$= \max(0, h_i - \max(h_{i(m(S))+1}, a_{m(S)})).$$

Therefore, if $m(S) < m(N)$, then

$$\begin{aligned} v(S) - v(S - \{i\}) &= \max(0, h_i - \max(h_{i(m(S))+1}, a_{m(S)})) \\ &\geqq \max(0, h_i - \max(a_{m(S)+1}, a_{m(S)})) \\ &= \max(0, h_i - a_{m(S)+1}) \geqq \max(0, h_i - a_{m(N)}) \end{aligned}$$

$$\geqq \max(0, h_i - \max h_{i(m(N))+1}, a_{m(N)}))$$
$$= v(N) - v(N - \{i\})$$

where the first inequality follows from the definition of $m(S)$, the second inequality follows from the monotone nondecreasingness of $a_k$ and the fact $m(S) < m(N)$, and the third inequality follows from the definition of $m(N)$. If $m(S) = m(N)$, then

$$\begin{aligned} v(S) - v(S - \{i\}) &= \max(0, h_i - \max(h_{i(m(S))+1}, a_{m(S)})) \\ &\geqq \max(0, h_i - \max(h_{m(S)+1}, a_{m(S)})) \\ &\geqq \max(0, h_i - \max(h_{m(N)+1}, a_{m(N)})) \\ &= v(N) - v(N - \{i\}) \end{aligned}$$

where the first inequality follows from the monotone nonincreasingness of $h_k$ and the fact $i(m(S)) + 1 \geqq m(S) + 1$, and the second inequality follows from the fact $m(S) = m(N)$. Hence B2** is satisfied, and thus the game $v$ is a strong big boss game.

Therefore the core $C(v)$ is given by $C(v) = \{x \in A(v): 0 \leqq x_i \leqq M_i(v) = h_i - \max(h_{m(N)+1}, a_{m(N)})$ for all $i$ with $2 \leqq i \leqq m(N)$, $x_i = 0$ for all $i$ with $i \geqq m(N) + 1\}$ (Theorem 3.2); this is exactly the same as the core given in Kaneko (1976, Theorem III). The $\tau$-value and the nucleolus are $\tau(v) = \nu(v) = (v(N) - (\sum_{i=2}^{m(N)} M_i(v))/2,\ M_2(v)/2, \ldots, M_{m(N)}(v)/2, 0, \ldots, 0)$ (Theorems 4.1 and 4.2). Figure 2 illustrates how the surplus (*i.e.*, the shaded area) is shared among the traders (the seller 1 and the buyers $2, \ldots, m(N)$) in the core, in the $\tau$-value and in the nucleolus for the case of $h_{m(N)+1} \geqq a_{m(N)}$. In each of these solutions, the lower part of the surplus (below the dotted line $h_{m(N)+1}$) always goes to the seller, and the upper part is shared by the seller and the buyers; in particular, in the $\tau$-value and the nucleolus, the seller gets exactly half of it (the doubly shaded area) and each buyer $i$, $i = 2, \ldots, m(N)$, gets half of $h_i - h_{m(N)+1}$. In the case of $n = 3$, $w = 1$ and $h_3 \geqq a_1$, we get $m(N) = 1$, $v(N) = h_2$, $M_2 = h_2 - h_3$ and $M_3 = 0$. Thus $C(v) = \{x \in A(v): 0 \leqq x_2 \leqq h_2 - h_3,\ x_3 = 0\}$ and $\tau(v) = ((h_2 + h_3)/2, (h_2 - h_3)/2, 0)$ follow as in von Neumann and Morgenstern (1953) and Tijs (1981). From Theorems 4.3, 4.5, and 4.6, we see that in the Shapley value the seller gets less and each of the buyers gets more compared with the payoffs in the $\tau$-value and in the nucleolus. Let $s$ be the largest integer such that $h_s \geqq a_1$. Then they get the same payoffs as in the $\tau$-value (and in the nucleolus) if and only if $w \geqq s$ and $a_1 = \ldots = a_s$, *i.e.*, the seller's utility function is essentially linear. In this case, we further see that the core is the stable core (Theorem 3.5).

### 5.2 A Bankruptcy Problem with One Big Claimant

A bankruptcy problem with a claimant set $N = \{1, 2, \ldots, n\}$ consists of $E$ and $(d_1, d_2, \ldots, d_n)$; $E \geqq 0$ is the estate which has to be divided among the claimants, and $d_i \geqq 0$ is the amount of the claim of $i$ for $i = 1, 2, \ldots, n$. Suppose $d_1 \geqq \ldots \geqq d_n$, and $E \leqq \sum_{i \in N} d_i$. A bankruptcy game is given by $v(S) = \max(E - \sum_{i \in N-S} d_i, 0)$ for all $S \subseteqq N$. The Shapley value, the nucleolus, and the $\tau$-value were studied in O'Neill (1982), Aumann and Maschler (1985), and Curiel, Maschler and Tijs (1986).

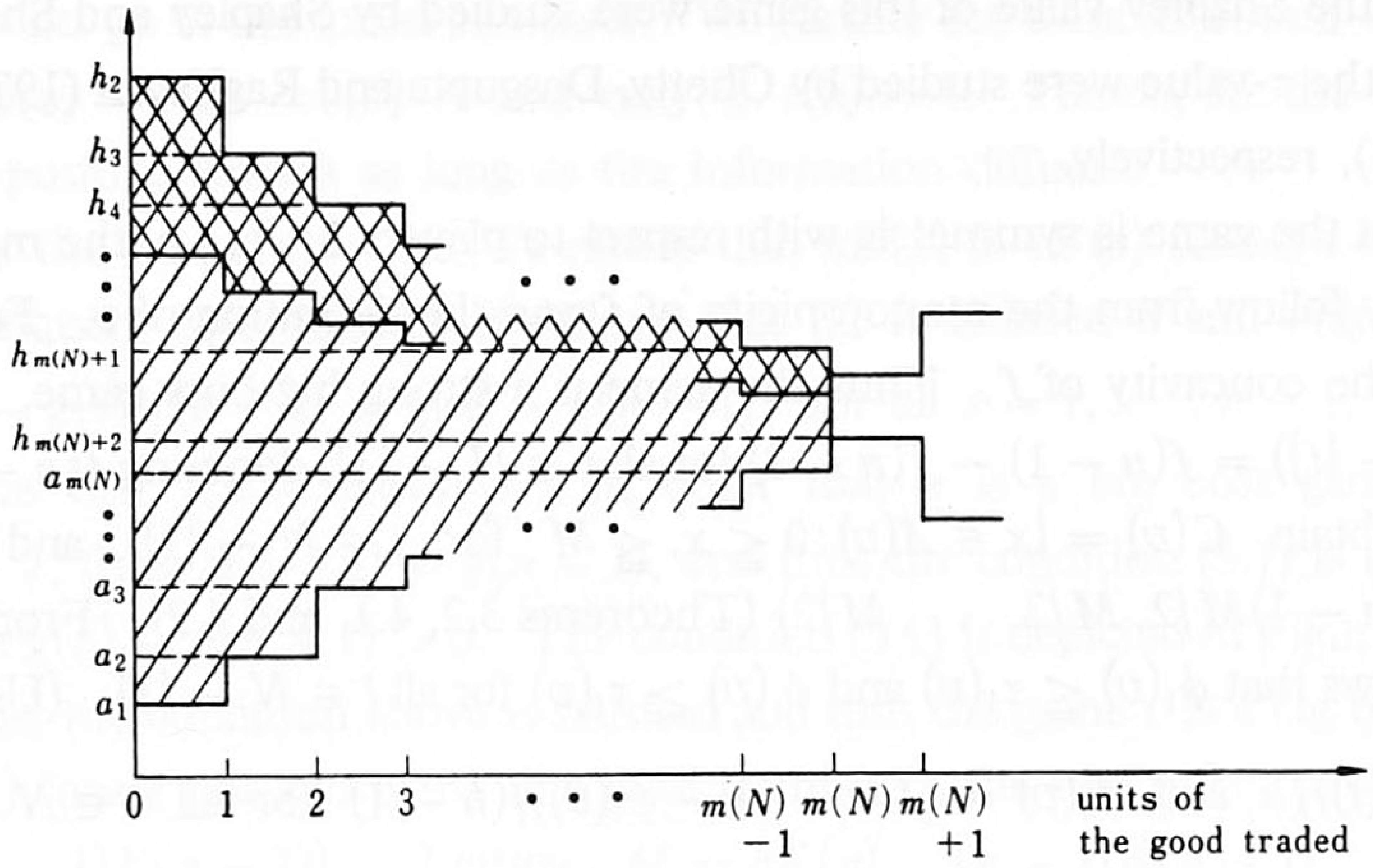


**Figure 2** The core, the $\tau$-value, and the nucleolus of the indivisible good market

In the following, we consider a class of bankruptcy games which have one big claimant. Namely, we assume $d_1 \geqq E$ and $\sum_{i \in N-\{1\}} d_i \leqq E$, and show that, under this condition, the bankruptcy games are strong big boss games. From the condition above, we get $v(S) = 0$ if $S \not\ni 1$, and $v(S) = E - \sum_{i \in N-S} d_i$ if $S \ni 1$. Therefore the monotonicity of $v$ and B1 easily follow. It easily follows further that B2** is satisfied in equalities. Therefore we see that the game $v$ is a strong big boss game, and further from Proposition 3.4 we notice that the game $v$ is convex. (The convexity of bankruptcy games generally holds. See Curiel, Maschler and Tijs (1986, Theorem 1)). Noting that $M_i(v) = d_i$ for all $i \in N - \{1\}$ and $v(N) = E$, we get $C(v) = \{x \in A(v): 0 \leqq x_i \leqq d_i$ for all $i \in N - \{1\}\}$, and $\phi(v) = \tau(v) = \nu(v) = (E - (\sum_{i \in N-\{1\}} d_i)/2, d_2/2, \ldots, d_n/2)$ (Theorems 3.2, 4.1, 4.2, and 4.5). For example, the third case of the Talmud (Aumann and Maschler (1985, p. 196)), in which $E = 300$, $d_1 = 300$, $d_2 = 200$, and $d_3 = 100$, satisfies the conditions above, and in that case $\phi(v) = \tau(v) = \nu(v) = (150, 100, 50)$. We further see from Theorem 3.5 that the core is the stable core.

### 5.3 A Production Economy with One Landowner and Many Peasants

We consider a production economy with one landowner and several landless peasants. Let $N = \{1, 2, \ldots, n\}$ be a set of agents where 1 is a landowner who cannot produce anything by himself, and $2, \ldots, n$ are landless peasants. The landowner 1 hires peasants to cultivate his land. The monetary value of the crop of the land depends only on the number of the peasants hired by the landowner, and is denoted by $f(t)$ where $t$ is the number of hired peasants. The real-valued function $f: \{0, 1, \ldots, n-1\} \longrightarrow R$ with $f(0) = 0$ is called a production function. $f$ is supposed to be monotone nondecreasing and concave, *i.e.*, $f(t) \geqq f(t-1)$ for all $t = 1, \ldots, n-1$, and $f(t+1) - f(t) \leqq f(t) - f(t-1)$ for all $t = 1, \ldots, n-2$. The characteristic function of this production economy is given by

$$v(S) = \begin{cases} f(|S| - 1) & \text{if } 1 \in S \\ 0 & \text{if } 1 \notin S. \end{cases}$$

The core and the Shapley value of this game were studied by Shapley and Shubik (1967); the nucleolus and the $\tau$-value were studied by Chetty, Dasgupta and Raghavan (1976) and Driessen and Tijs (1984), respectively.

Noticing that the game is symmetric with respect to players $2, \dots, n$, the monotonicity of $v$ and B1 directly follow from the monotonicity of $f$ and the definition of $v$. Further B2** also follows from the concavity of $f$. Thus the game is a strong big boss game. Since $M_i(v) = v(N) - v(N - \{i\}) = f(n-1) - f(n-2)$ for all $i \in N - \{1\}$, denoting $f(n-1) - f(n-2)$ by $M$, we obtain $C(v) = \{x \in A(v) : 0 \leqq x_i \leqq M$ for $i \in N - \{1\}\}$ and $\tau(v) = \nu(v) = (f(n-1) - (n-1)M/2, M/2, \dots, M/2)$ (Theorems 3.2, 4.1, and 4.2). From Theorems 4.3 and 4.6, it follows that $\phi_1(v) \leqq \tau_1(v)$ and $\phi_i(v) \geqq \tau_i(v)$ for all $i \in N - \{1\}$. (Using the formula $\phi_1(v) = (\sum_{t=0}^{n-1} f(t))/n$ and $\phi_i(v) = (f(n-1) - \phi_1(v))/(n-1)$ for all $i \in N - \{1\}$ given in Shapley and Shubik (1967), one may directly examine these inequaities.) We further see from Theorem 4.5 that $\phi_1(v) = \tau_1(v) = \nu_1(v)$ (and $\phi(v) = \tau(v) = \nu(v)$) holds only when $f$ is linear. In this case, the core is the stable core (Theorem 3.5).

**5.4 A Market of an Information Good with Symmetric Externalities**

We consider a market of an information good consisting of a trader set $N = \{1, 2, \dots, n\}$ and two kinds of goods; an information good and money. An information good is initially owned only by trader 1, and traders $2, \dots, n$ are demanders of the information. The monetary value of the information good, *i.e.*, the profit gained by utilizing the information, depends only on the number of its possessors. Let $E(s)$ be the value of the information for each possessor when the information is shared by $s$ traders. We suppose the value of information never increases as the number of its possessors increases, but the possessors can gain a positive profit even if the information is shared by all traders. Hence $E(1) \geqq \dots \geqq E(n) > 0$. Nonpossessors of the information are supposed to gain no profit. Supposing there exists no perfect patent protection, Muto (1986) and Nakayama (1986) studied how the information is traded and diffused in this market.

In the following, assuming a perfect patent protection on the information, we analyze this market from the cooperative game theoretic viewpoint. If the perfect patent protection is present, the characteristic function of this market is given by

$$v(S) = \begin{cases} \max\{tE(t) : 1 \leqq t \leqq |S|\} & \text{if } 1 \in S \\ 0 & \text{if } 1 \notin S. \end{cases}$$

The monotonicity of $v$ and B1 easily follow from the definition of $v$. We examine a condition under which the game $v$ satisfies B2 and B2**. Noting that for every $S \subseteqq N$ with $S \ni 1$, $v(S)$ depends only on $|S|$, we denote $v(S)$ by $g(s)$ in case $S \ni 1$ and $|S| = s$. For each $s = 1, 2, \dots, n$, let $l(s)$ be the number such that $v(s) = l(s)E(l(s))$. For simplicity, we assume that $l(s)$ is uniquely determined for each $s = 1, 2, \dots, n$.

First consider the case $l(n) \leqq n - 1$. In this case we have $g(n) - g(n-1) = 0$, and thus, from the monotonicity of $v$, B2 and B2** are satisfied, *i.e.*, $v$ is a strong big boss game. Therefore the core $C(v)$ consists of the single imputation $\{(l(n)E(l(n)), 0, \dots, 0)\}$; this is the $\tau$-value and also the nucleolus (Theorems 3.2, 4.1, and 4.2). Hence the information is shared by $l(n)$ traders and all of the profits go to the initial possessor. We further see from Theorem 4.5 and Proposition 3.4 that $\phi(v) = \tau(v) = \nu(v)$ if and only if $l(n) = 1$. Hence, in the Shapley value, demanders get positive payoffs as long as the information diffuses.

Now consider the case $l(n) = n$. We notice that $nE(n) > sE(s)$ for all $s = 1, \ldots, n-1$ since $l(n)$ is uniquely determined. It follows that B2 is satisfied if and only if

(5.1) $(g(n) - g(s))/(n - s) \geqq g(n) - g(n-1)$ for all $s = 1, \ldots, n-1$.

We easily notice that $l(n-1) = n-1$ in order that $v$ is a big boss game. In fact, if $l(n-1) \leqq n-2$, then $g(n-1) = g(n-2)$, and thus the condition (5.1) is not satisfied for $s = n-2$ since $g(n) - g(n-1) > 0$. The condition (5.1) is depicted in Figure 3. From this figure we see that the condition above is satisfied and thus the game $v$ is a big boss game if and only if all $sE(s)$ locate not above the line passing through the two points $A = (n, E(n))$ and $B = (n-1, (n-1)E(n-1))$. Letting $M = nE(n) - (n-1)E(n-1)$, we get from Theorems 3.2, 4.1, and 4.2, $C(v) = \{x \in A(v) : 0 \leqq x_i \leqq M$ for all $i \in N - \{1\}\}$ and $\tau(v) = \nu(v) = (nE(n) - (n-1)M/2, M/2, \ldots, M/2)$. The information is shared by all traders and, in the $\tau$-value and the nucleolus, each of the demanders gets exactly half of its contribution to the grand coalition. It follows from Theorem 4.5 that the Shapley value coincides with the $\tau$-value (and the nucleolus) when and only when all $sE(s)$ are on the line passing through the two points $A$ and $B$. In this case, the core is the stable core (Theorem 3.5). Since the game is symmetric with respect to the demanders $2, \ldots, n$, if at least one $sE(s)$ is below the line, each of the demanders gets more in the Shapley value (Corollary 4.4).

An economically plausible example would be the case in which the marginal returns to coalition sizes are nonincreasing; namely,

$$sE(s) - (s-1)E(s-1) \geqq (s+1)E(s+1) - sE(s) \quad \text{for all } s = 2, 3, \ldots, n-1.$$

That the necessary and sufficient condition (5.1) for B2 is satisfied follows immediately. We further see that under this condition, B2* (or B2**) is also satisfied and thus the game is a strong big boss game.

We hereupon make a brief remark on the relation between this model and the production

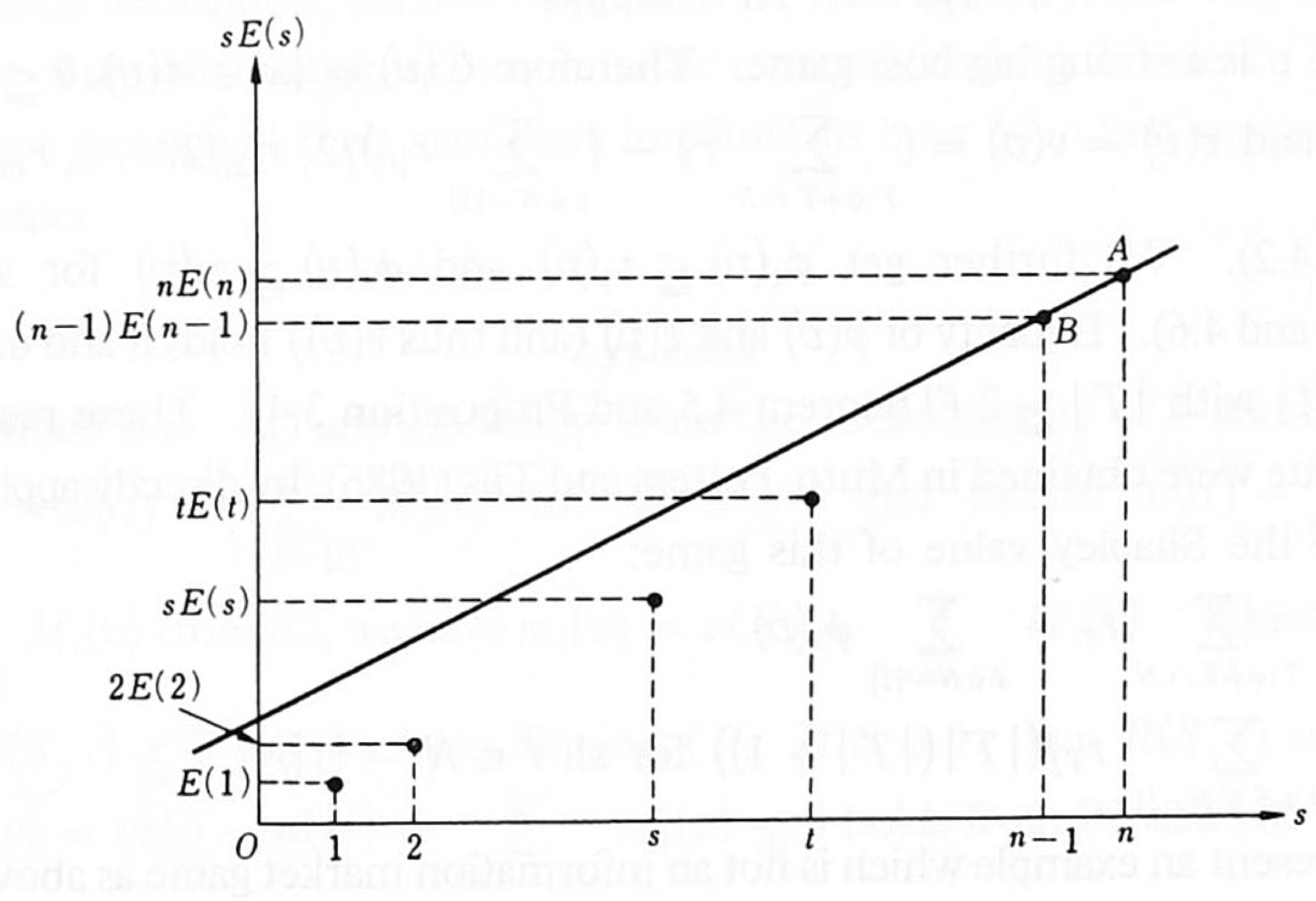


**Figure 3** An illustration of the condition (5.1)

economy analyzed in the previous subsection. Throughout the analysis above, we have assumed that the initial possessor of information can gain a profit by utilizing the information. Even if he cannot gain any profit by himself, one may define, a similar game

$$v(S)=\begin{cases}\max\{tE(t):1\leqq t\leqq |S|-1\} & \text{if } 1\in S\\ 0 & \text{if } 1\notin S,\end{cases}$$

where for each $s=1,\ldots,n-1$, $E(s)$ denotes the value of the information for each informed demander in case the information is diffused to $s$ demanders, and may develop a similar analysis as above. Here if we let $f(s)=sE(s)$ for all $s=1,\ldots,n-1$ and $f(0)=0$, then we get a production economy with a more general production function $f$; neither monotone nondecreasing nor concave. One may analyze this more general production economy in a similar manner as above.

**5.5 A Market of an Information Good Concerning a New Product**

We consider a market consisting of a set $N=\{1,2,\ldots,n\}$ of firms and two kinds of goods; an information good with respect to a manufacturing process of a new product, and money. Firm 1 has the information, and each of the firms $2,\ldots,n$ may start producing the product once it gets the information. The market of the new product is partitioned into $\{M_T\}_{\phi\neq T\subseteqq N}$, where $M_T$ denotes the market into which only the firms belonging to $T$ can enter. For each $M_T$, the profit $r_T\geqq 0$ gained by selling this product in $M_T$ is evaluated. Supposing a perfect patent protection on the information is provided, the characteristic function of this information market is given by

$$v(S)=\begin{cases}\sum_{T:T\cap S\neq\phi} r_T & \text{if } 1\in S\\ 0 & \text{if } 1\notin S.\end{cases}$$

This game is called an information market game and studied by Muto, Potters and Tijs (1986).

We show that this game is a strong big boss game. The monotonicity of $v$ and B1 easily follow from the definition of $v$. Further, since $M_i(v)=v(N)-v(N-\{i\})=r_{\{i\}}$ for all $i\in N-\{i\}$ and $v(S)-v(S-\{i\})=\sum_{T:T\cap S\neq\phi} r_T-\sum_{T:T\cap(S-\{i\})\neq\phi} r_T\geqq r_{\{i\}}$ for all $S\subseteqq N$, B2** is satisfied. Thus the game $v$ is a strong big boss game. Therefore $C(v)=\{x\in A(v):0\leqq x_i\leqq r_{\{i\}}$ for all $i\in N-\{1\}\}$, and $\tau(v)=\nu(v)=(\sum_{T:\phi\neq T\cap N} r_T-(\sum_{i\in N-\{1\}} r_{\{i\}})/2,\ r_{\{2\}}/2,\ldots,r_{\{n\}}/2)$ (Theorems 3.2, 4.1, and 4.2). We further get $\phi_1(v)\leqq\tau_1(v)$ and $\phi_i(v)\geqq\tau_i(v)$ for all $i\in N-\{1\}$ (Theorems 4.3 and 4.6). Equality of $\phi(v)$ and $\tau(v)$ (and thus $\nu(v)$) holds if and only if $r_T=0$ for all $T\subseteqq N-\{1\}$ with $|T|\geqq 2$ (Theorem 4.5 and Proposition 3.4). These results concerning the Shapley value were obtained in Muto, Potters and Tijs (1986), by directly applying the following formula of the Shapley value of this game:

$$\phi_1(v)=\sum_{T:\phi\neq T\cap N} r_T-\sum_{i\in N-\{1\}}\phi_i(v)$$

and $\phi_i(v)=\sum_{T:i\in T\subseteqq N-\{1\}} r_T/(|T|(|T|+1))$ for all $i\in N-\{1\}$.

Finally we present an example which is not an information market game as above but a big boss game.

**Example 5.1:** $N = \{1,2,3,4\}$, $v(N) = 2$, $v(S) = 2$ if $1 \in S$ and $|S| = 3$, $v(S) = 1$ if $1 \in S$ and $|S| = 2$, and $v(S) = 0$ for all other $S$.
It is easy to see that this game is a strong big boss game. But this is not an information game. In fact, since $v(N) = v(S) = 2$ for all $S \subsetneqq N$ with $|S| = 3$, there is no nonnegative numbers $\{r_T\}_{\phi \neq T \subseteqq \{1,2,3,4\}}$ which satisfy $v(S) = \sum_{T: T \cap S \neq \phi} r_T$ for all $S \subseteqq N$.

## 6. Concluding Remarks

In this paper, we have studied a class of monotonic characteristic function form games satisfying the two conditions B1: $v(S) = 0$ if $S \not\ni 1$ and B2: $v(N) - v(S) \geqq \sum_{i \in N-S} (v(N) - v(N - \{i\})$ if $S \ni 1$. A game in this class was called a big boss game and the player 1 was called a big boss. We have principally shown that in big boss games: (1) the core is a parallelotope in the sense that the core imputation of each weak player $2, \ldots, n$ is bounded by 0 and his marginal contribution to the grand coalition; (2) the core is itself a subsolution; (3) the core is a stable set if and only if the game is convex; in big boss games the convexity implies that $v(S) = v(\{1\}) + \sum_{i \in S-\{1\}} M_i(v)$ for all coalitions $S$ containing the big boss 1; (4) the $\tau$-value coincides with the nucleolus and both are the center of the core; (5) the Shapley value of the big boss 1 is generally below the $\tau$-value and the nucleolus, and they are equal if and only if the game is convex; in this case they are equal also for weak players $2, \ldots, n$; (6) if the condition B2*: $v(T) - v(S) \geqq \sum_{i \in T-S} (v(N) - v(N - \{i\})$ for all $1 \in S \subseteqq T$, which is stronger than B2, holds true, *i.e.*, the game is a strong big boss game, then the Shapley value of each of the weak players is above the $\tau$-value and the nucleolus.

Further we have shown that several economic games earlier discussed in the literature, such as an indivisible good market with one seller and many buyers, a bankruptcy problem with one big claimant, a production economy with one landowner and many peasants, and with a concave production function, and a market of an information with respect to a new product with one initial possessor and many demanders, fall into this class. We have further made clear under what conditions a market of an information with symmetric externalities is in this class. The relations of solutions of these games and their economic implications have been first captured comprehensively in this paper.

## Appendix

**Proof of Proposition 3.1:** (1) and (2) follow from the monotonicity of $v$ and B1, respectively. Since $R(S,1) = v(S) - \sum_{i \in S-\{1\}} M_i(v)$ for all $S \ni 1$ and further $v(S) - \sum_{i \in S-\{1\}} M_i(v) \leqq v(N) - \sum_{i \in N-\{1\}} M_i(v)$ from B2, we have $m_1(v) = v(N) - \sum_{i \in N-\{1\}} M_i(v)$. Take $i \in N - \{1\}$. If $S \not\ni 1$, then $R(S,i) \leqq 0$ follows from B1 and (1). If $S \ni 1$, then $R(S,i) = v(S) - M_1(v) - \sum_{j \in S-\{1,i\}} M_j(v) = v(S) - v(N) - \sum_{j \in S-\{1,i\}} M_j(v) \leqq 0$ holds from B2 and (1). Since $R(\{i\},i) = v(\{i\}) = 0$ holds from B1, we obtain $m_i(v) = 0$, and thus (3) follows. Finally, since $v(N)$

$-\sum_{i\in N-\{1\}} M_i(v) \geqq v(\{1\})$ holds from B2, $M(v) \geqq m(v) \geqq 0$ follows from the monotonicity of $v$, (1), (2) and (3). (Q.E.D.)

**Proof of Proposition 3.3:** $H(v) \supseteqq CC(v)$ is trivial since $m_i(v) = 0$ for all $i \in N - \{1\}$ (Proposition 3.1(3)). To show the reverse inclusion, take $x \in H(v)$. It suffices to show that $m_1(v) \leqq x_1 \leqq M_1(v)$. $x_1 \leqq M_1(v)$ is trivial since $M_1(v) = v(N)$ (Proposition 3.1(2)). Since $m_1(v) = v(N) - \sum_{i\in N-\{1\}} M_i(v)$ (Proposition 3.1(3)) and $x_i \leqq M_i(v)$ for all $i \in N - \{1\}$, we have

$$x_1 = v(N) - \sum_{i\in N-\{1\}} x_i \geqq v(N) - \sum_{i\in N-\{1\}} M_i(v) = m_1(v). \quad \text{(Q.E.D.)}$$

**Proof of Proposition 3.4:** (1)$\Rightarrow$(2): Take $S \subseteqq N$ with $S \ni 1$. If $S = N$, then (2) is trivial. Suppose $S \subsetneqq N$ and let $N - S = \{i(1), \ldots, i(n-s)\}$ where $n - s$ is the number of players in $N - S$. Then from the convexity of $v$ we obtain $v(S \cup \{i(1), \ldots, i(t-1), i(t)\}) - v(S \cup \{i(1), \ldots, i(t-1)\}) \leqq M_{i(t)}(v)$ for all $t = 1, \ldots, n-s$. Equality holds in case $t = n - s$. Summing up all these inequalities, we get $v(N) - v(S) \leqq \sum_{i\in N-S} M_i(v)$. Thus from B2 the desired equality follows.

(2)$\Rightarrow$(3): Take $S \subseteqq N$ with $S \ni 1$. If $S = N$, then taking $\{1\}$ as $S$ in (2), we obtain $v(N) = v(\{1\}) + \sum_{i\in N-\{1\}} M_i(v)$ as desired. Suppose $S \subsetneqq N$. Then from (2) we have $v(S) = v(N) - \sum_{i\in N-S} M_i(v)$. From these two equalities, we obtain the desired equality.

(3)$\Rightarrow$(4): Take $S \subseteqq N$ with $S \supsetneqq \{1\}$ and take $i \in S - \{1\}$. Then from (3) we have $v(S) = v(\{1\}) + \sum_{i\in S-\{1\}} M_i(v)$ and $v(S - \{i\}) = v(\{1\}) + \sum_{i\in S-\{1,i\}} M_i(v)$. Hence the desired equality follows.

(4)$\Rightarrow$(1): Take $i \in N$ and take $S, T \subseteqq N$ with $i \in S \subseteqq T$. First suppose $i = 1$. Then since from B1 we have $v(S - \{i\}) = v(T - \{i\}) = 0$, the desired inequality follows from the monotonicity of $v$. Now suppose $i \neq 1$. If $1 \notin S$, then $v(S) - v(S - \{i\}) = 0$ and thus the desired inequality follows again from the monotonicity of $v$. If $1 \in S$, then from (4) we have $v(S) - v(S - \{i\}) = M_i(v) = v(T) - v(T - \{i\})$. Hence $v$ is convex. (Q.E.D.)

**Proof of Proposition 4.6:** Suppose a game $v \in G^N$ satisfies B2*. Take $R \subseteqq N$ with $R \supsetneqq \{1\}$, and take a player $i \neq 1$ in $R$. Letting $T = R$ and $S = R - \{i\}$ in B2*, we get $v(R) - v(R - \{i\}) \geqq M_i(v)$ as desired. Conversely suppose $v$ satisfies B2**. Take two sets $S, T \subseteqq N$ such that $1 \in S \subseteqq T$. In case $S = T$, B2* clearly holds, and thus we suppose $S \subsetneqq T$. Let $T - S = \{i(1), \ldots, i(t-s)\}$ where $t, s$ are the number of players in $T, S$, respectively. Then from B2** we obtain $v(S \cup \{i(1), \ldots, i(r-1), i(r)\}) - v(S \cup \{i(1), \ldots, i(r-1)\}) \geqq M_{i(r)}(v)$ for all $r = 1, \ldots, t-s$. Summing up all these inequalities, we obtain $v(T) - v(S) \geqq \sum_{i\in T-S} M_i(v)$ as desired. (Q.E.D.)

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